Notes in Analysis for Economists

Pantelis Karapanagiotis

October 6, 2017

Contents

Chapter 1

Metric Spaces

1.1 Metrics and Norms

Definition 1.1. [Metric]
Let $X \neq \emptyset$. A function $\rho\colon X \times X \to \mathbb{R}$ is called metric on X if

1. $\rho \geq 0$ and $\rho(x, y) = 0$ if and only if $x = y$ (non-negativity),
2. $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$ (symmetry) and
3. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$ (triangular inequality).

The pair (X, ρ) is called metric space. When the metric is understood from context, we refer to metric spaces only using the set X.

Example 1.2. 1. Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$. This metric is called the usual metric on $\mathbb{R}$.

2. Let $X = \mathbb{R}^n$. The functions

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| \qquad \forall x, y \in \mathbb{R}^n$$

and

$$d_\infty(x, y) = \max_{i=1,\dots,n} |x_i - y_i| \qquad \forall x, y \in \mathbb{R}^n$$

define metrics on $\mathbb{R}^n$.

3. Let X be an arbitrary non empty set. Then

$$\delta(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

is a (trivial) metric on X. This metric is called the discrete metric on X.

Proposition 1.3. *For any (X, ρ) metric space we have*

$$|\rho(x, z) - \rho(z, y)| \leq \rho(x, y) \qquad \forall x, y, z \in X.$$

Proof. We use the triangular inequality two times. In particular

$$\rho(x,z) \leq \rho(x,y) + \rho(y,z) \implies \rho(x,z) - \rho(y,z) \leq \rho(x,y)$$

and

$$\rho(z,y) \leq \rho(z,x) + \rho(x,y) \implies \rho(z,y) - \rho(z,x) \leq \rho(x,y).$$

Then, by symmetry we get the result. □

Definition 1.4. [Relative metric]
Let (X,ρ) be a metric space and $\emptyset \neq A \subseteq X$. The restriction of ρ on $A \times A$, i.e.

$$\rho_A\colon A \times A \to \mathbb{R}\colon (x,y) \mapsto \rho_A(x,y) = \rho(x,y),$$

is called the relative metric on A induced by ρ.

Remark 1.5. Every non-empty subset A of a metric space (X,ρ) constitutes a metric space endowed with the relative metric ρ_A.

Definition 1.6. [Norm]
Let X be a real vector space. A function $\|\cdot\| : X \to \mathbb{R}$ is called norm on X if

1. $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0_X$ (non-negativity),
2. $\|\lambda x\| = |\lambda| \, \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$ (positive homogeneity) and
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y, z \in X$ (triangular inequality).

The pair $(X, \|\cdot\|)$ is called normed space.

Example 1.7. 1. Let $X = \mathbb{R}$. The absolute value $|\cdot|$ is a norm on $\mathbb{R}$.

2. Let $X = \mathbb{R}^n$. The functions

$$\|x\|_1 = \sum_{j=1}^{n} |x_j| \qquad \forall x \in \mathbb{R}^n.$$

and

$$\|x\|_\infty = \max_{i=1,\ldots,n} |x_i| \qquad \forall x \in \mathbb{R}^n$$

are norms on on $\mathbb{R}^n$.

Proposition 1.8. ***[Metrics induced by norms]***
Let $(X, \|\cdot\|)$ be a normed space. Then

$$d(x,y) = \|x - y\| \qquad \forall x, y \in X$$

is a metric on X. Moreover, d is translation invariant and positive homogeneous.

Proof. Let $x, y, z \in X$ and $\lambda \in \mathbb{R}$. We have $d(x,y) = \|x-y\| \geq 0$. Furthermore

$$0 = d(x,y) \iff 0 = \|x-y\| \iff x-y=0 \iff x=y,$$

hence d is non-negative. For symmetry, notice that

$$d(x,y) = \|x-y\| = |-1|\,\|y-x\| = d(y,x).$$

Triangular inequality of d follows from the corresponding property of $\|\cdot\|$. Specifically

$$d(x,y) = \|x-y\| = \|x-z+z-y\| \leq \|x-z\| + \|z-y\| = d(x,z) + d(z,y).$$

We also show the additional properties of d. Note that

$$d(x+z, y+z) = \|x+z-y-z\| = \|x-y\| = d(x,y),$$

i.e. d is translation invariant. Lastly,

$$d(\lambda x, \lambda y) = \|\lambda(x-y)\| = |\lambda|\,\|x-y\| = |\lambda|\,d(x,y),$$

which proves positive homogeneity. □

Remark 1.9. The definition of normed spaces requires that the underlying set of the space has a linear structure. Note that we need this structure for both property 2 and 3 of the norm. This linear structure leads to the extra properties of the metrics that are induced by norms.

Proposition 1.10. ***[Hölder inequality]***
Let $p, q \in (1,\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for all $x, y \in \mathbb{R}^n$ we have

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}.$$

Proof. If $x_i = 0$ for all $i = 1, \ldots, n$, then inequality of the proposition holds with equality. Similarly for the case that $y_i = 0$ for all $i = 1, \ldots, n$. Now, suppose that the above two cases do not hold. Then $\sum_{i=1}^{n} |x_i|^p$ and $\sum_{i=1}^{n} |y_i|^q$ are positive. Define

$$a_i = \frac{|x_i|}{\left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}}}$$

and

$$b_i = \frac{|y_i|}{\left(\sum_{j=1}^{n} |y_j|^q\right)^{\frac{1}{q}}}$$

Fix $i = 1, \ldots, n$. If $a_i b_i = 0$, then clearly

$$a_i b_i \leq \frac{1}{p} a_i^p + \frac{1}{q} b_i^q.$$

Suppose that $a_i, b_i > 0$. Since log is a concave function, we have

$$\log\left(\frac{1}{p} a_i^p + \frac{1}{q} b_i^q\right) \geq \frac{1}{p} \log a_i^p + \frac{1}{q} \log b_i^q = \log(a_i b_i)$$

and by monotonicity of log, we conclude

$$a_i b_i \le \frac{1}{p} a_i^p + \frac{1}{q} b_i^q.$$

Therefore

$$\sum_{i=1}^n a_i b_i \le \frac{1}{p} \sum_{i=1}^n a_i^p + \frac{1}{q} \sum_{i=1}^n b_i^q = \frac{1}{p} + \frac{1}{q} = 1,$$

which is equivalent to the inequality that is to be shown. □

Remark 1.11. In the context of of proposition (1.10), we refer to pairs p, q such that $\frac{1}{p} + \frac{1}{q} = 1$ as conjugate exponents.

Corollary 1.12. ***[Cauchy-Schwarz inequality]***
For all $x, y \in \mathbb{R}^n$ we have

$$\sum_{i=1}^n |x_i y_i| \le \left(\sum_{i=1}^n |x_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^n |y_i|^2 \right)^{\frac{1}{2}}.$$

Proposition 1.13. ***[Minkowski inequality]***
Let $p \in (1, \infty)$. Then, for all $x, y \in \mathbb{R}^n$ we have

$$\left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{1}{p}} \le \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}}.$$

Proof. We prove the proposition by applying Hölder's inequality. If $\sum_{i=1}^n |x_i + y_i|^p = 0$ then the inequality holds. Suppose that the last sum is positive. Let $q = \frac{p}{p-1}$ and note that p, q are conjugate exponents. We have

$$\begin{aligned}
\sum_{i=1}^n |x_i + y_i|^p &= \sum_{i=1}^n |x_i + y_i|^{p-1} |x_i + y_i| \\
&\le \sum_{i=1}^n |x_i + y_i|^{p-1} |x_i| + \sum_{i=1}^n |x_i + y_i|^{p-1} |y_i| \\
&\le \left(\sum_{i=1}^n |x_i + y_i|^{q(p-1)} \right)^{\frac{1}{q}} \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |x_i + y_i|^{q(p-1)} \right)^{\frac{1}{q}} \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \\
&= \left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{1}{q}} \left[\left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \right].
\end{aligned}$$

Therefore

$$\left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{1}{p}} = \left(\sum_{i=1}^n |x_i + y_i|^p \right)^{1-\frac{1}{q}} \le \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}},$$

as needed. □

Example 1.14. We can now give some more examples of norms and metrics.

1. Let $X = \mathbb{R}^n$ and $p > 1$. We can define the norm

$$\|x\|_p = \left(\sum_{j=1}^{n} |x_j|^p \right)^{\frac{1}{p}} \qquad \forall x \in \mathbb{R}^n.$$

The norms of the family $\|\cdot\|_p$ for $p \in [1, \infty]$ are called p-norms. For $p = 2$ we get the Euclidean norm on $\mathbb{R}^n$. We denote the normed space $(\mathbb{R}^n, \|\cdot\|_p)$ with ℓ_p^n.

2. Let $X = \mathbb{R}^n$ and $p > 1$. The function

$$d_p(x, y) = \left(\sum_{j=1}^{n} |x_j - y_j|^p \right)^{\frac{1}{p}} \qquad \forall x, y \in \mathbb{R}^n$$

is a metric on $\mathbb{R}^n$. For $p = 2$ we get the Euclidean metric on $\mathbb{R}^n$.

Definition 1.15. [Bounded spaces and sets]

(i) A metric space (X, ρ) is called bounded if there exists some $M > 0$ such that

$$\sup_{x,y \in X} \rho(x, y) < M.$$

We then say that X has diameter $\sup_{x,y \in X} \rho(x, y)$ and write

$$\operatorname{diam}(X) = \sup_{x,y \in X} \rho(x, y).$$

(ii) We say that a subset $\emptyset \neq A \subseteq X$ is bounded if the metric space (A, ρ_A) is bounded. We conventionally agree that $\operatorname{diam}(\emptyset) = 0$.

Example 1.16. 1. Let $n \in \mathbb{N}$ and $p \in [1, \infty]$. The metric space $(\mathbb{R}^n, d_p)$ is not bounded.

2. The metric space $(\mathbb{R}, \rho)$ where $\rho(x, y) = |\arctan x - \arctan y|$ is bounded. Moreover $\operatorname{diam}(\mathbb{R}, \rho) = \pi$.

3. Any non-empty set endowed with the discrete metric is bounded.

1.2 Sequence spaces

Definition 1.17. [Sequences]
Let (X, ρ) be a metric space. A function $x \colon \mathbb{N} \to X$ is called sequence. To simplify notation, we write $x_n = x(n)$. A sequence x is also written as $(x_n)_{n=1}^{\infty}$, or $(x_n)_n$, or simply (x_n).

Definition 1.18. [Convergent sequences]
Let (X, ρ) be a metric space and $(x_n)_n$ a sequence in X. We say that $(x_n)_n$ converges to $x \in X$ with respect to the metric ρ if for every $\varepsilon > 0$ there exists a $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that if $n \geq n_0$ then $\rho(x_n, x) < \varepsilon$. We then write $x_n \xrightarrow{\rho} x$, or $x_n \to x$.

Proposition 1.19. *Let $(x_n)_n$ be a sequence in (X, ρ). Then, $x_n \to x$ if and only if $(\rho(x_n, x))_n$ is a null sequence in $(\mathbb{R}, d_1)$, that is, if and only if $\rho(x_n, x) \xrightarrow{d_1} 0$.*

Proof. Observe that for all $\varepsilon > 0$ we have $\rho(x_n, x) < \varepsilon$ if and only if $|\rho(x_n, x) - 0| < \varepsilon$. We then apply the definition of convergence. □

Proposition 1.20. ***[Limits are unique]***
Let $(x_n)_n$ be a sequence in (X, ρ). If $(x_n)_n$ is convergent, then its limit is unique.

Proof. Suppose that $(x_n)_n$ has two limits, namely $x, y \in X$. Let $\varepsilon > 0$. Since $x_n \to x$ there exists some $n_1 \in \mathbb{N}$ such that $\rho(x, x_n) < \frac{\varepsilon}{2}$ for all $n \geq n_1$. Similarly, by $x_n \to y$ there exists some $n_2 \in \mathbb{N}$ such that $\rho(x_n, y) < \frac{\varepsilon}{2}$ for all $n \geq n_2$. Let $n_0 = \max(n_1, n_2)$. Then for all $n \geq n_0$ we have

$$\rho(x, y) \leq \rho(x, x_n) + \rho(x_n, y) < \varepsilon$$

and since ε was arbitrary, we conclude that $\rho(x, y) = 0$. □

Definition 1.21. [Basic sequences]
Let (X, ρ) be a metric space. We say that a sequence $\{x_n\}_n \subseteq X$ is basic or Cauchy if for every $\varepsilon > 0$ there exists a $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that if $m, n \geq n_0$ then $\rho(x_m, x_n) < \varepsilon$ holds.

Proposition 1.22. ***[Convergent sequences are basic]***
Let (X, ρ) be a metric space. If $\{x_n\}_n \subseteq X$ is convergent then it is basic.

Proof. Suppose that $x_n \to x$ and let $\varepsilon > 0$. Choose $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $\rho(x_n, x) < \frac{\varepsilon}{2}$. Then for all $n, m \geq n_0$ we have

$$\rho(x_n, x_m) \leq \rho(x_n, x) + \rho(x, x_m) < \varepsilon.$$

□

Definition 1.23. [Bounded sequences]
Let (X, ρ) be a metric space. We say that a sequence $\{x_n\}_n \subset X$ is bounded if there exists a $M > 0$ such that $\rho(x_m, x_n) < M$ for all $m, n \in \mathbb{N}$.

Proposition 1.24. ***[Basic sequences are bounded]***
Let (X, ρ) be a metric space. If $\{x_n\}_n \subseteq X$ is basic then it is bounded.

Proof. Suppose that (x_n) is a basic. There exists a $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\rho(x_n, x_{n_0}) < 1$. Let

$$M = \max\{\rho(x_1, x_{n_0}), \ldots, \rho(x_{n_0-1}, x_{n_0}), 1\}.$$

Let $n, m \in \mathbb{N}$. Then by the triangular inequality we have

$$\rho(x_n, x_m) \leq \rho(x_n, x_{n_0}) + \rho(x_{n_0}, x_m) < 2M.$$

□

Corollary 1.25. ***[Convergent sequences are bounded]***
Let (X, ρ) be a metric space. If $\{x_n\}_n \subseteq X$ is convergent then it is bounded.

Example 1.26. We consider examples of sequences in $\mathbb{R}$.

1. The sequence $(\frac{1}{n})_n$ converges to zero with respect to the usual metric in $\mathbb{R}$. Hence it is basic and also bounded. For instance $\left|\frac{1}{n} - \frac{1}{m}\right| < 1$ for all $n, m \in \mathbb{N}$.

2. The sequence $(\log \frac{1}{n})_n$ is basic with respect to the metric $\rho(x,y) = |e^x - e^y|$. However it is not convergent. Suppose that $\log \frac{1}{n} \to x$. Then

$$\left|\frac{1}{n} - e^x\right| = \rho\left(\log \frac{1}{n}, x\right) \to 0.$$

The last limit implies that $\frac{1}{n} \xrightarrow{d_1} e^x > 0$, which is a contradiction.

3. The sequence $((-1)^n)_n$ is bounded by 2 with respect to the usual metric, but it is not basic.

Definition 1.27. [Supremum and infimum]
Let A be a non-empty subset of $\mathbb{R}$. An element $s \in \mathbb{R}$ is called least upper bound or supremum of A if

1. it is an upper bound of A, i.e. for all $a \in A$ we have $a \leq s$, and
2. if $b \in \mathbb{R}$ is an upper bound of A, then $b \geq s$.

We write $s = \sup A$ or $s = \sup_{a\in A} a$. An element $l \in \mathbb{R}$ is called greatest lower bound or infimum of A if

1. it is a lower bound of A, i.e. for all $a \in A$ we have $a \geq l$, and
2. if $b \in \mathbb{R}$ is a lower bound of A, then $b \leq l$.

Similarly, we write $l = \inf A = \inf_{a\in A} a$.

Remark 1.28. For every non-empty A subset of $\mathbb{R}$ we have $supA = -\inf(-A)$.

Proposition 1.29. ***[ε characterization of supremum]***
Let $\emptyset \neq A \subseteq \mathbb{R}$ and s be an upper bound of A. Then s is the supremum of A if and only if for all $\varepsilon > 0$ there exists $a \in A$ such that $a > s - \varepsilon$.

Proof. $\Longrightarrow$: Suppose that for some $\varepsilon > 0$ we have $a \leq s - \varepsilon$ for all $a \in A$. Then $s - \varepsilon$ is an upper bound of A and $s - \varepsilon < s$, hence s is not the supremum of A.
$\Longleftarrow$: Suppose that s in an upper bound of A but not the supremum. Then there exists an upper bound $s' \in \mathbb{R}$ such that $s' < s$. Let $\varepsilon > 0$ such that $s' < s - \varepsilon$. Then for all $a \in A$ we have $a \leq s'$ and hence $a < s - \varepsilon$. □

Corollary 1.30. ***[ε characterization of infimum]***
Let $\emptyset \neq A \subseteq \mathbb{R}$ and l be a lower bound of A. Then l is the infimum of A if and only if for all $\varepsilon > 0$ there exists $a \in A$ such that $a < l + \varepsilon$.

Proposition 1.31. ***[Bolzano-Weierstrass theorem]***
Let $(x_n)_n$ be a bounded sequence in $\mathbb{R}^k$ for $k \geq 1$. Then there exists a converging subsequence.

Proof. We first show the proposition for $k = 1$. Define the set of indices of peaks of the sequence, namely

$$A = \{n \in \mathbb{N} \colon\ x_n > x_m \quad \forall m > n\}.$$

If A is empty, then $(x_n)_n$ has an increasing subsequence. If A is finite, let N be its maximal element. Then the subsequence $(x_n)_{n>N}$ has no peaks and hence we can choose an increasing subsequence. If A is infinite we can pick an increasing $(n_i)_i$ in A so that the subsequence $(x_{n_i})_i$ is decreasing. Therefore, in any case we can find a monotone subsequence of $(x_n)_n$.

Let $(x_{n_k})_k$ be a monotone subsequence. We will show that if $(x_{n_k})_k$ is increasing and bounded, then it converges. Similar arguments are applicable for the decreasing case. The set $\{x_{n_k}\}_k$ is non empty and bounded. Hence its supremum, say s, exists. Let $\varepsilon > 0$. Then there exists some $x_{n_{k_0}}$ such that $s > x_{n_{k_0}} > s - \varepsilon$. Since $(x_{n_k})_k$ is increasing, if $k \geq k_0$ we also have $s > x_{n_k} > s - \varepsilon$. Thus, for all $k \geq k_0$ we have $|x_{n_k} - s| < \varepsilon$, as needed.

Now suppose that $k > 1$. Since $(x_n)_n$ is bounded, there exists some $M > 0$ such that

$$|x_n(1)| \leq \left(\sum_{j=1}^{k} |x_n(j)|^2 \right)^{\frac{1}{2}} \leq M \qquad \forall n \in \mathbb{N}.$$

Hence $(x_n(1))_n$ is bounded and so there exists a $M_1 \subseteq \mathbb{N}$ such that $(x_n(1))_{n \in M_1}$ is a convergent subsequence. Say that $x_n(1) \xrightarrow{n \in M_1} x(1)$. We repeat the argument for $(x_n(2))_n$ and find $M_2 \subseteq M_1$ such that $x_n(2) \xrightarrow{n \in M_2} x(2)$. Observe that by construction we also have $x_n(1) \xrightarrow{n \in M_2} x(1)$. Using induction, we find $M_k \subseteq \mathbb{N}$ such that $x_n(i) \xrightarrow{n \in M_k} x(i)$ for all $i = 1, \ldots, k$. We claim that $x_n \to x$. Let $\varepsilon > 0$. For each $i = 1, \ldots, k$ pick n_i such that $|x_n(i) - x(i)| < \frac{\varepsilon}{2\sqrt{k}}$. Set $n_0 = \max\{n_1, \ldots, n_k\}$. Then for all $n \geq n_0$ we have

$$\left(\sum_{j=1}^{k} |x_n(j) - x(j)|^2 \right)^{\frac{1}{2}} \leq \left(k \frac{\varepsilon^2}{4k} \right)^{\frac{1}{2}} < \varepsilon \qquad \forall n \in \mathbb{N}.$$

as needed. □

Definition 1.32. [**ℓ_p spaces**]
Let $p \in [1, \infty]$ and denote

$$X = \begin{cases} \left\{ x \colon \mathbb{N} \to \mathbb{R} \quad : \quad \sum_{i=1}^{\infty} |x_i|^p < \infty \right\} & \text{if } p < \infty \\ \left\{ x \colon \mathbb{N} \to \mathbb{R} \quad : \quad \sup_{i \in \mathbb{N}} |x_i| < \infty \right\} & \text{if } p = \infty. \end{cases}$$

For $p \in [1, \infty)$ we refer to elements of X as p-sumable sequences. For each $x \in X$ the function

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} & \text{if } p < \infty \\ \sup_{i \in \mathbb{N}} |x_i| & \text{if } p = \infty. \end{cases}$$

is a norm on X. We denote the normed spaces $(X, \|\cdot\|_p)$ by ℓ_p.

Definition 1.33. [**Subsequential limits**]
Let $(x_n)_n$ be a sequence in $\mathbb{R}$. We say that $s \in X$ is the greatest subsequential limit of $(x_n)_n$ if

$$s = \lim_{n \to \infty} \sup_{m \geq n} x_m.$$

In such a case we write $\limsup_{n \to \infty} x_n = s$. We say that $l \in X$ is the least subsequential limit of $(x_n)_n$ if

$$l = \lim_{n \to \infty} \inf_{m \geq n} x_m.$$

and write $\liminf_{n \to \infty} x_n = l$.

Proposition 1.34. *Let $(x_n)_n$ be a sequence in $\mathbb{R}$. If $s = \limsup x_n$ then there exists a subsequence of $(x_n)_n$ that converges to s.*

Proof. Define $y_n = \sup_{m \geq n} x_m$ for all $n \in \mathbb{N}$. Note that if $n > m$, then $\{x_k\}_{k \geq m} \supseteq \{x_k\}_{k \geq n}$, hence $y_m \geq y_n$, that is $(y_k)_k$ is decreasing. We have $y_n \to s$. Hence there is some $n_1 \in \mathbb{N}$ such that $0 < y_{n_1} - s < 1$. Similarly, there exists some $n_1 < n_2 \in \mathbb{N}$ such that $0 < y_{n_2} - s < \frac{1}{2}$. Recursively, we construct a subsequence $(y_{n_k})_k$ such that $0 < y_{n_k} - s < \frac{1}{k}$ for all $k \in \mathbb{N}$. By the ε characterization of supremum, there exists some $n_{k_1} \geq n_1$ such that

$$y_{n_1} = \sup_{m \geq n_1} x_m \geq x_{n_{k_1}} \geq \sup_{m \geq n_1} x_m - 1 = y_{n_1} - 1.$$

Furthermore, there exists some $n_{k_2} \geq \max\{n_{k_1}, n_2\}$ such that $y_{n_2} \geq x_{n_{k_2}} \geq y_{n_2} - \frac{1}{2}$. Recursively, we construct a subsequence $(x_{n_{k_m}})_m$ such that $y_{n_m} \geq x_{n_{k_m}} \geq y_{n_m} - \frac{1}{m}$ for all $m \in \mathbb{N}$. Thus

$$s - \frac{1}{m} < y_{n_m} - \frac{1}{m} \leq x_{n_{k_m}} \leq y_{n_m}$$

for all $m \in \mathbb{N}$. Therefore $x_{n_{k_m}} \to s$. ☐

Corollary 1.35. *Let $(x_n)_n$ be a sequence in $\mathbb{R}$. If $l = \liminf x_n$ then there exists a subsequence of $(x_n)_n$ that converges to l.*

1.3 Function spaces

Definition 1.36. [Boundedness]
Let $f\colon (X, \rho) \to (Y, d)$ be function between two metric spaces. We say that f is bounded on X if the metric space $(f(X), \rho_{f(X)})$ is bounded. Equivalently, f is bounded if there exists some $M > 0$ such that for all $x_1, x_2 \in X$ we have $d(f(x_1), f(x_2)) < M$.

Definition 1.37. [Continuity]
Let $f\colon (X, \rho) \to (Y, d)$ be function between two metric spaces. We say that f is continuous at $x_0 \in X$, if for all $\varepsilon > 0$ there exists some $\delta > 0$ such that for all $x \in X$ with $\rho(x, x_0) < \delta$ we have $d(f(x), f(x_0)) < \varepsilon$. We say that f is continuous on X if it is continuous at every $x \in X$.

Remark 1.38. The above definition implies that a function $f\colon (X, \rho) \to (Y, d)$ is continuous, if for all $x_0 \in X$ and for all $\varepsilon > 0$ there exists some $\delta = \delta(x_0, \varepsilon) > 0$ such that for all $x_1 \in X$ with $\rho(x_1, x_0) < \delta$ we have $d(f(x_1), f(x_0)) < \varepsilon$.

Example 1.39. We consider real functions of a real variable with the usual metric on $\mathbb{R}$.

1. The function $f\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}\colon f(x) = \frac{1}{x}$ is continuous but not bounded.

2. The function $f(x) = 1_{\mathbb{R}_{\geq 0}}(x)$ is not continuous but bounded.

3. The function $f(x) = \sin x$ is continuous and bounded.

4. The function $f(x) = 1_{\mathbb{R}_{\geq 0}}(x)e^x$ is neither continuous, nor bounded.

Proposition 1.40. ***[Sequential continuity]***
Let $f\colon (X, \rho) \to (Y, d)$ be function between two metric spaces. The function f is continuous at $x_0 \in X$ if and only if it is sequentially continuous at x_0, that is for all $\{x_n\}_n \subseteq X$ such that $x_n \xrightarrow{\rho} x_0$ we have $f(x_n) \xrightarrow{d} f(x_0)$.

Proof. $\implies$: Suppose that f is continuous at x_0 and let $\varepsilon > 0$, to show that $d(f(x_n), f(x_0)) < \varepsilon$ finally holds. By continuity, there exists some $\delta > 0$ such that for all $x \in X$ with $\rho(x, x_0) < \delta$ we have $d(f(x), f(x_0)) < \varepsilon$. Since $x_n \to x$, there exists some n_0 such that for all $n \geq n_0$ we have $\rho(x_n, x_0) < \delta$. Then for all $n \geq n_0$ we get $d(f(x_n), f(x_0)) < \varepsilon$, as needed.

$\impliedby$: Assume that for any sequence $(x_n)_n$ that converges in x_0, we have $f(x_n) \xrightarrow{d} f(x_0)$, to show that f is continuous. Suppose that this is not the case. Then there exists some $\varepsilon > 0$ such that for all $\delta > 0$ there exists some $x \in X$ with $\rho(x, x_0) < \delta$ for which $d(f(x), f(x_0)) > \varepsilon$. In particular, for each $n \in \mathbb{N}$, there exists some $x_n \in X$ such that $\rho(x_n, x_0) < \frac{1}{n}$ and $d(f(x_n), f(x_0)) > \varepsilon$. By construction $x_n \to x_0$ and thus $d(f(x_n), f(x_0)) > \varepsilon$ contradicts our original assumption. $\square$

Definition 1.41. [Uniform Continuity]
Let $f\colon (X, \rho) \to (Y, d)$ be function between two metric spaces. We say that f is uniformly continuous, if for all $\varepsilon > 0$ there exists some $\delta = \delta(\varepsilon) > 0$ such that for all $x_0, x_1 \in X$ with $\rho(x_1, x_0) < \delta$ we have $d(f(x_1), f(x_0)) < \varepsilon$.

Remark 1.42. Comparing the definitions of continuity and uniform continuity reveals that every uniformly continuous function is also continuous. The other direction is not true. Uniform continuity is stricter than continuity. Observe that in the uniform continuity definition the choice of δ does not depend on x_0. This means that the same δ works for all the points in the domain of X.

Example 1.43. Consider the function $f(x) = e^x$ with the usual metric on $\mathbb{R}$. One can show that f is continuous. Let $x_0 \in \mathbb{R}$ and $\varepsilon > 0$. Let $0 < \bar{\varepsilon} < \min\{\varepsilon, e^{x_0}\}$. Choose

$$\delta < \min\left\{\log\left(1 + \bar{\varepsilon}e^{-x_0}\right), -\log\left(1 - \bar{\varepsilon}e^{-x_0}\right)\right\}.$$

Then for all $x \in X$ such that $|x - x_0| < \delta$ we have

$$-\delta < x - x_0 < \delta \iff e^{-\delta + x_0} < e^x < e^{\delta + x_0}.$$

From the right hand side, we get

$$e^x < e^{\delta + x_0} < \left(1 + \bar{\varepsilon}e^{-x_0}\right)e^{x_0} \implies e^x - e^{x_0} < \varepsilon$$

and from the left hand side

$$e^x > e^{-\delta + x_0} > \left(1 - \bar{\varepsilon}e^{-x_0}\right)e^{x_0} \implies e^x - e^{x_0} > -\varepsilon.$$

Combining the above inequalities we conclude that $|e^x - e^{x_0}| < \varepsilon$.

However, f is not uniformly continuous. Towards contradiction, suppose that f is uniformly continuous. Then for $\varepsilon = 1$, there exists some $\delta > 0$ such that for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$ we have $|e^x - e^y| < 1$. By monotonicity, we have

$$e^{\frac{\delta}{2}} - 1 > 0.$$

Since $e^x \to \infty$ for $x \to \infty$ there exists some $x_0 \in X$ such that

$$e^{x_0}\left(e^{\frac{\delta}{2}} - 1\right) > 1.$$

Choose $y_0 = x_0 + \frac{\delta}{2}$. Clearly, $|x_0 - y_0| < \delta$ and

$$|e^{x_0} - e^{y_0}| = e^{x_0}\left|1 - e^{\frac{\delta}{2}}\right| > 1.$$

Definition 1.44. [Lipschitz Continuity]
Let $f\colon (X,\rho) \to (Y,d)$ be a function between two metric spaces. We say that f is Lipschitz continuous with modulus $L > 0$, if for all all $x_1, x_2 \in X$ we have

$$d(f(x_1), f(x_2)) < L\rho(x_1, x_2).$$

Proposition 1.45. *If $f\colon (X,\rho) \to (Y,d)$ is Lipschitz continuous, then it is uniformly continuous (and hence continuous).*

Proof. Suppose that f is Lipschitz continuous with modulus L. Let $\varepsilon > 0$ and pick $\delta < \frac{\varepsilon}{L}$. Then for all $\rho(x_1, x_2) < \delta$ we get

$$d(f(x_1), f(x_2)) < L\rho(x_1, x_2) < \varepsilon.$$

□

Example 1.46. We give some examples of basic function spaces. Let (X,ρ) and (Y,d) be two metric spaces.

1. Denote

$$\mathcal{B}(X,Y) = \{f\colon X \to Y \quad : \ f \text{ is bounded}\}.$$

If Y is a normed space, say with norm $\|\cdot\|$, then the function

$$\|f\|_\infty = \sup_{x\in X} \|f(x)\| \qquad \forall f \in \mathcal{B}(X,Y)$$

defines a norm on $\mathcal{B}(X,Y)$. The norm $\|\cdot\|_\infty$ is called supremum norm. The pair $(\mathcal{B}(X,Y), \|\cdot\|_\infty)$ becomes a normed space. When $Y = \mathbb{R}$ endowed with the usual metric we simply write $\mathcal{B}(X)$ instead of $\mathcal{B}(X,\mathbb{R})$.

2. Denote

$$\mathcal{C}(X,Y) = \{f\colon X \to Y \quad : \ f \text{ is continuous}\}.$$

Suppose that Y is a normed space. As continuous functions can be unbounded, $\|\cdot\|_\infty$ may not be defined for all function in $\mathcal{C}(X,Y)$. If we set

$$\mathcal{C}_b(X,Y) = \mathcal{C}(X,Y) \cap \mathcal{B}(X,Y),$$

then $\|\cdot\|_\infty$ is well defined on $\mathcal{C}_b(X,Y)$ and the pair $(\mathcal{C}_b(X,Y), \|\cdot\|_\infty)$ also becomes a normed space. Specifically $\mathcal{C}_b(X,Y)$ is a subspace of $\mathcal{B}_b(X,Y)$. Again, for $Y = \mathbb{R}$ endowed with the usual metric we simply write $\mathcal{C}(X)$ and $\mathcal{C}_b(X)$.

3. Similarly, we define

$$\mathcal{U}(X,Y) = \{f\colon X \to Y \quad : \ f \text{ is uniformly continuous}\}$$

and, for Y being a normed space, the the normed space $\mathcal{U}_b(X,Y)$ endowed with the supremum norm.

1.4 Topology of metric spaces

Definition 1.47. [Open and closed balls]
Let (X, ρ) be a metric space. Let $x \in X$ and $r > 0$. We denote

$$B_r(x) = \{y \in X : \rho(x, y) < r\}$$

and call $B_r(x)$ the open ball with center x and radius r. We write

$$\bar{B}_r(x) = \{y \in X : \rho(x, y) \leq r\}$$

and say that $\bar{B}_r(x)$ is the closed ball with center x and radius r.

Definition 1.48. [Open and closed sets]
Let (X, ρ) be a metric space and $A \subseteq X$. We say that A is open (with respect to the metric ρ) if for all $x \in A$ there exists some $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$. We say that A is closed if its complement, that is $X \backslash A$, is open.

Proposition 1.49. *Let (X, ρ) be a metric space, $x \in X$ and $r > 0$. Then*

(i) $B_r(x)$ is open and

(ii) $\bar{B}_r(x)$ is closed.

Proof. (i) Let $y \in B_r(x)$ and choose $0 < \varepsilon < r - \rho(x, y)$. Let $z \in B_\varepsilon(y)$. Then

$$\rho(x, z) \leq \rho(x, y) + \rho(y, z) < r.$$

Since z was arbitrary, we conclude $B_\varepsilon(y) \subseteq B_r(x)$.
(ii) Let $y \in X \backslash \bar{B}_r(x)$ and choose $0 < \varepsilon < \rho(x, y) - r$. Let $z \in B_\varepsilon(y)$. Then

$$\rho(x, z) \geq |\rho(x, y) - \rho(y, z)| = \rho(x, y) - \rho(y, z) > r.$$

Hence $X \backslash \bar{B}_r(x)$ is open. □

Proposition 1.50. *Let (X, ρ) be a metric space and $\{A_i\}_{i \in I}$ a collection of open sets in X. Then*

(i) $\cup_{i \in I} A_i$ is open and

(ii) if I is finite, then $\cap_{i \in I} A_i$ is open.

Proof. (i) Fix $x \in \cup_{i \in I} A_i$. There exists some $i_0 \in I$ such that $x \in A_{i_0}$. Since A_{i_0} is open, there exists some $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A_{i_0}$. Then, we also have

$$B_\varepsilon(x) \subseteq A_{i_0} \subseteq \cup_{i \in I} A_i.$$

(ii) Fix some $x \in \cap_{i \in I} A_i$. Let $j \in I$. Since A_j is open, there exists some $\varepsilon_j > 0$ such that $B_{\varepsilon_j}(x) \subseteq A_j$. We repeat the argument for every $j \in I$ and find a collection $(\varepsilon_j)_{j \in I}$ such that $\varepsilon_j > 0$ and $B_{\varepsilon_j}(x) \subseteq A_j$. Set $\varepsilon = \min_{j \in I} \varepsilon_j$. Note that ε is well defined and positive because I is finite. Then, for every $j \in I$ we have

$$B_\varepsilon(x) \subseteq B_{\varepsilon_j}(x) \subseteq A_j,$$

from which we conclude that

$$B_\varepsilon(x) \subseteq \cap_{i \in I} A_i.$$

□

Corollary 1.51. *Let (X,ρ) be a metric space and $\{A_i\}_{i\in I}$ a collection of closed sets in X. Then*

(i) $\cap_{i\in I}A_i$ *is closed and*

(ii) if I is finite, then $\cup_{i\in I}A_i$ *is closed.*

Proposition 1.52. ***[Sequential characterizations of closed and open sets]***
Let (X,ρ) be a metric space and $A\subseteq X$. Then

(i) A is open if and only if for all $x\in A$ and for all sequences $(x_n)_n$ in X with $x_n\to x$ there exists some $n_0\in\mathbb{N}$ such that if $n\geq n_0$ then $x_n\in A$ and

(ii) A is closed if and only if for all sequences $(x_n)_n$ in A with $x_n\to x$ we have $x\in A$.

Proof. (i) $\Longrightarrow$: Suppose that A is open and let $x\in A$ and some sequence $(x_n)_n$ that converges to x. Since A is open, there exists some $\varepsilon>0$ for which we have $B_\varepsilon(x)\subseteq A$. Since $x_n\to x$, there exists some $n_0\in\mathbb{N}$ such that if $n\geq n_0$ then $\rho(x_n,x)<\varepsilon$. In other words, if $n\geq n_0$, then $x_n\in B_\varepsilon(x)\subseteq A$.
$\Longleftarrow$: Suppose that A is not open. Then we can find a point $x\in A$ such that for all $n\in\mathbb{N}$ we have $B_{\frac{1}{n}}(x)\backslash(X\backslash A)\neq\emptyset$. We recursively construct a sequence $(x_n)_n$ such that for all $n\in\mathbb{N}$ we have $x_n\in B_{\frac{1}{n}}(x)\backslash A$. By construction $x_n\to x$.
(ii) $\Longrightarrow$: Let A be a closed subset of X. Suppose that there exists a sequence $(x_n)_n$ in A with $x_n\to x$ and $x\in X\backslash A$. The last set is open and so, there exists an $\varepsilon>0$ such that $B_\varepsilon(x)\subseteq X\backslash A$. Since $x_n\to x$ we can find a $n_0\in\mathbb{N}$ such that for all $n\geq n_0$ we have $x_n\in B_\varepsilon(x)$. This is a contradiction.
$\Longleftarrow$: Suppose that A is not closed. Then $X\backslash A$ is not open. Hence, by (i), there exists some $x\in X\backslash A$ and a sequence $(x_n)_n$ in X that converges to x and for all $n\in\mathbb{N}$ there exists an $m>n$ such that $x_m\notin X\backslash A$. Thus, we can choose a subsequence of $(x_{n_m})_m$ of $(x_n)_n$ such that $x_{n_m}\in A$ for all $m\in\mathbb{N}$. Since $(x_n)_n$ converges to x, also $(x_{n_m})_m$ converges to x. ☐

Definition 1.53. [Limit point of a set]
Let (X,ρ) be a metric space and $A\subseteq X$. We say that $x\in X$ is a limit point of A, if for every $\varepsilon>0$, we have $B_\varepsilon(x)\cap A\backslash\{x\}\neq\emptyset$. The set of all limit points of A is called derived set and is denoted by A'.

1.5 Exercises

A Group

A.1. Let X be a non empty set and $f\colon X \to \mathbb{R}$ be an injection. Show that $\rho(x, y) = |f(x) - f(y)|$ is a metric on X.

A.2. (a) Show that for all $p \in [1, \infty]$, the function

$$\|x\|_p = \begin{cases} \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} & p \in [1, \infty) \\ \max_{i=1,\dots,n} |x_i| & p = \infty \end{cases} \qquad \forall x \in \mathbb{R}^n.$$

defines a norm on $\mathbb{R}^n$.

(b) Show that for all $p \in [1, \infty]$, the function

$$d_p(x, y) = \|x - y\|_p \qquad \forall x, y \in \mathbb{R}^n$$

defines a metric on $\mathbb{R}^n$.

A.3. Let X be a non empty set and $\delta\colon X \times X \to \mathbb{R}$ the discrete metric.

(a) Show that (X, δ) is a metric space.

(b) Show that a sequence in X is convergent if and only if is finally constant.

(c) Let ρ be the Euclidean metric on $\mathbb{R}$. Show that all functions $f\colon (X, \delta) \to (\mathbb{R}, \rho)$ are continuous.

A.4. Show that $(\mathbb{R}, \rho)$ where $\rho(x, y) = |\arctan x - \arctan y|$ is a bounded metric space. In particular it holds that $\operatorname{diam}(\mathbb{R}, \rho) = \pi$.

A.5. Prove corollary (1.35).

A.6. Prove corollary (1.51).

A.7. Let $(X, \|\cdot\|)$ be a normed space. Show that a non empty subset A of X is bounded if and only if there exists $M > 0$ such that for all $x \in A$ we have $\|x\| \leq M$.

B Group

B.1. Let (X, ρ) be a metric space. Show that a sequence $(x_n)_n$ in X converges to some point $x \in X$ if and only if every subsequence converges to x.

B.2. Let $f\colon (X, \rho) \to (Y, d)$ be function between two metric spaces. Show that f is uniformly continuous if and only if for all $(x_n^1)_n$, $(x_n^2)_n$ in X such that $\rho(x_n^1, x_n^2) \to 0$ we have $d(f(x_n^1), f(x_n^2)) \to 0$.

B.3. Let (X, ρ) be a metric space and $(Y, \|\cdot\|)$ a normed space. Let $f, g\colon X \to Y$ and $\lambda \in \mathbb{R}$. Show that if f and g are

(a) bounded, then $f + g$ and λf are bounded,

(b) continuous, then $f + g$ and λf are continuous,

(c) uniformly continuous, then $f + g$ and λf are uniformly continuous.

B.4. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable and that its derivative is bounded. Show that f is Lipschitzt continuous.

C Group

C.1. We consider the spaces ℓ_p with $p \in [1, \infty]$.

(a) Show that if $1 \leq p \leq q \leq \infty$, then $\ell_p \subseteq \ell_q$. Moreover, if $p < q$ then the inclusion is strict.

(b) Show that if $x \in \ell_p$ for $p < \infty$, then x is converging to zero.

(c) Find a sequence that converges to zero and does not belong in ℓ_p for all $p \in [1, \infty)$.

(d) Find a sequence that belongs in $(\cup_{p>1} \ell_p) \setminus \ell_1$.

Chapter 2

Completeness

2.1 Completeness

Definition 2.1. [Complete metric spaces]
A metric space (X, ρ) is called complete if every basic sequence converges with respect to ρ to some point in X.

Remark 2.2. We take completeness of $\mathbb{R}$ with respect to the usual metric as known. See Dedekind cuts for a construction.

Proposition 2.3. ***[Supremum/infimum characterization of Completeness]***
Let X be a non-empty and bounded subset of $\mathbb{R}$ endowed with the usual metric. Then X is complete if and only if it contains the least upper and greatest lower bounds of every $\emptyset \neq A \subseteq X$.

Proof. $\implies$: Suppose that X is complete. Let A be a non empty subset of X and denote with s its supremum. From the ε characterization of supremum we can construct a sequence $(a_n)_n$ in A such that $a_n \to s$. As convergent, $(a_n)_n$ is basic in $\mathbb{R}$. Hence it is also basic in X and by completeness we conclude that $s \in X$. Similarly we show that the infimum of A belongs in X.
$\impliedby$: Suppose that X contains the least upper and greatest lower bounds of its subsets. Let $(x_n)_n$ be a basic sequence in X. As basic, $(x_n)_n$ is bounded so we can pick a monotone subsequence $(x_{n_k})_k$. Without loss of generality, suppose that $(x_{n_k})_k$ is increasing. Analogous arguments can be made when it is decreasing. Since $(x_{n_k})_k$ is basic in X, it is also basic in $\mathbb{R}$, so it converges. Let s be its limit. Note that

$$|x_n - s| \leq |x_n - x_{n_k}| + |x_{n_k} - s|$$

implies that $(x_n)_n$ also converges to x. It remains to show that x belongs in X. Since $(x_{n_k})_k$ is increasing, we have

$$s = \lim_k x_{n_k} = \sup_k x_{n_k}.$$

Let $A = \{x_{n_k}\}_k$ and note that $A \subseteq X$. Hence $s = \sup_k \{x_{n_k}\} \in X$. Since $(x_n)_n$ was an arbitrary basic sequence, we conclude that X is complete. □

Remark 2.4. [Supremum and maximum]
Note that for every bounded and complete X, we can apply the previous theorem using $A = X$. This implies that $\sup X \in X$. Therefore, when X is bounded and complete, its maximum exists. We argue similarly for the minimum.

Remark 2.5. [Completeness is not a topological property]
The metric space $((0,1),d_1)$ is not complete. For example, the sequence $(\frac{1}{n+1})_n$ is basic, because it is convergent in $(\mathbb{R},d_1)$, but it is not convergent in $(0,1)$. On the other hand, the metric space $((0,1),d)$, where

$$d(x,y) = |x-y| + \left|\frac{1}{x} - \frac{1}{y}\right| + \left|\frac{1}{1-x} - \frac{1}{1-y}\right|,$$

is complete. Both metrics induce the same topology.

Proposition 2.6. *[ℓ_p^k completeness]*
For every $k > 1$ and $p \in [1,\infty]$, the space ℓ_p^k is complete.

Proof. Fix $k > 1$ and $p \in [1,\infty)$. Let $(x_n)_n$ be a basic sequence in ℓ_p^k. Let $\varepsilon > 0$. Since $(x_n)_n$ is basic there exists some $n_0 \in \mathbb{N}$ such that for all $m,n > n_0$ we have

$$\sum_{j=1}^{k} |x_n(j) - x_m(j)|^p \leq \varepsilon^p.$$

Thus, for each $i = 1,\ldots,k$ we get

$$|x_n(i) - x_m(i)|^p \leq \sum_{j=1}^{k} |x_n(j) - x_m(j)|^p \leq \varepsilon^p,$$

which implies that the sequence $(x_n(i))_n$ is basic. Since $\mathbb{R}$ is complete, $(x_n(i))_n$ converges to some $x(i) \in \mathbb{R}$. Define $x = (x(j))_{j=1}^k$. We will show that $x_n \xrightarrow{d_p} x$. Since $(x_n(1))_n$ converges to $x(1)$, there exists some n_1 such that for all $n \geq n_1$ we have

$$|x(1) - x_n(1)| < \varepsilon k^{-\frac{1}{p}}.$$

We repeat the above argument for every $i = 2,\ldots,k$ and find n_i such that for all $n \geq n_i$ we have

$$|x(i) - x_n(i)| < \varepsilon k^{-\frac{1}{p}}.$$

Set $\bar{n} = \max_{i=1,\ldots,k} n_i$. Then

$$\left(\sum_{i=1}^{k} |x(i) - x_n(i)|^p\right)^{\frac{1}{p}} < \left(k\varepsilon^p \frac{1}{k}\right)^{\frac{1}{p}} = \varepsilon$$

for all $n \geq \bar{n}$.

Now, let $p = \infty$ and $\varepsilon > 0$. With argumentation similar to the one of the previous case we have

$$|x_n(i) - x_m(i)| \leq \sup_{j=1,\ldots,k} |x_n(j) - x_m(j)| \leq \varepsilon,$$

for all $i = 1,\ldots,k$. By completeness of $\mathbb{R}$ we find $(x(i))_{i=1}^k$ such that

$$|x_n(i) - x(i)| < \varepsilon,$$

for all $i = 1,\ldots,k$. The result follows by taking the supremum in both sides of the last inequality. □

Proposition 2.7. ***[ℓ_p completeness]***
For every $p \in [1, \infty]$, the space ℓ_p is complete.

Proof. Fix $p \in [1, \infty)$ and $\varepsilon > 0$. Let $(x_n)_n$ be a basic sequence in ℓ_p. Then, for n, m large enough we have

$$\|x_n - x_m\|_p < \varepsilon.$$

Hence, for any $k \in \mathbb{N}$ we get

$$|x_n(k) - x_m(k)| \leq \|x_n - x_m\|_p < \varepsilon,$$

i.e. $(x_n(k))_n$ is basic. Let $x(k)$ be its limit. Since this holds for all $k \in \mathbb{N}$, we can define $x = (x(k))_k$. We have to show that $x \in \ell_p$ and that $x_n \to x$.

Pick some $N \in \mathbb{N}$. By Minkoskwi inequality, we have

$$\left(\sum_{k=1}^{N} |x(k)|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{N} |x(k) - x_n(k)|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{N} |x_n(k)|^p\right)^{\frac{1}{p}}.$$

For n large enough, as in the proof of proposition (2.6), we have

$$\left(\sum_{k=1}^{N} |x(k)|^p\right)^{\frac{1}{p}} \leq \varepsilon + \left(\sum_{k=1}^{N} |x_n(k)|^p\right)^{\frac{1}{p}}.$$

Letting $N \to \infty$ we conclude

$$\|x\|_p \leq \varepsilon + \|x_n\|_p,$$

so x is in ℓ_p.

Lastly, we show convergence. Pick again $N \in \mathbb{N}$. We have

$$\left(\sum_{k=1}^{N} |x(k) - x_n(k)|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{N} |x_m(k) - x_n(k)|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{N} |x_m(k) - x(k)|^p\right)^{\frac{1}{p}}.$$

For m large enough the second term in the right hand side becomes arbitrarily small, i.e.

$$\left(\sum_{k=1}^{N} |x(k) - x_n(k)|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{N} |x_m(k) - x_n(k)|^p\right)^{\frac{1}{p}} + \frac{\varepsilon}{2}.$$

Taking the limit for $N \to \infty$ gives

$$\|x - x_n\|_p \leq \|x_n - x_m\|_p + \frac{\varepsilon}{2}$$

and since $(x_n)_n$ is basic, the first term becomes arbitrarily small for n, m large enough. Thus

$$\|x - x_n\|_p \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

A similar strategy is used to prove the result for $p = \infty$. □

Definition 2.8. **[Convergence of functions]**
Let (X,ρ) be a metric space, $f_n\colon X\to\mathbb{R}$ for all $n\in\mathbb{N}$ be a sequence of functions and $f\colon X\to\mathbb{R}$. We say that $(f_n)_n$ converges pointwise to f if for all $x\in X$, the sequence $(f_n(x))_n$ converges to $f(x)$. In such a case, we write $x_n \xrightarrow{pw} x$. We say that $(f_n)_n$ converges uniformly to f if $d_\infty(f,f_n)\to 0$ and we write $x_n \xrightarrow{u} x$.

Proposition 2.9. ***[Completeness of bounded function spaces]***
Let (X,ρ) be a metric space. The metric space $(\mathcal{B}(X),d_\infty)$ is complete.

Proof. Let $(f_n)_n$ be a basic sequence in $\mathcal{B}(X)$. Fix $x\in X$ and let $\varepsilon>0$. There exists $n_0\in\mathbb{N}$ such that

$$|f_n(x)-f_m(x)|\le \|f_n-f_m\|_\infty<\varepsilon,$$

so the sequence $(f_n(x))_n$ is basic in $(\mathbb{R},d_1)$. As basic, the sequence $(f_n(x))_n$ converges to some point y_x in $\mathbb{R}$. Define $f(x)=y_x$. Since limits are unique, $f\colon X\to\mathbb{R}$ is well defined. Observe that

$$\begin{aligned}|f(x)| &= |f(x)-f_n(x)+f_n(x)|\\ &\le |f(x)-f_n(x)|+|f_n(x)|\\ &\le |f(x)-f_n(x)|+\|f_n\|_\infty\,,\end{aligned}$$

hence for n large enough we get $|f(x)|\le 1+\|f_n\|_\infty$, which implies that $f\in\mathcal{B}(X)$. Lastly, we will show that $\|f-f_n\|_\infty\to 0$ as $n\to\infty$. Fix $x\in X$ and let $\varepsilon>0$. There exists a $n_0\in\mathbb{N}$ such that if $n,m\ge n_0$, then

$$\begin{aligned}|f(x)-f_n(x)| &\le |f(x)-f_m(x)|+|f_m(x)-f_n(x)|\\ &\le |f(x)-f_n(x)|+\|f_m-f_n\|_\infty\\ &< \frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\frac{\varepsilon}{2}.\end{aligned}$$

Since $x\in X$ was arbitrary we get

$$\sup_{x\in X}|f(x)-f_n(x)|=\|f-f_n\|_\infty\le\frac{\varepsilon}{2}<\varepsilon.$$

□

Proposition 2.10. ***[Complete subspaces]***
Let (X,ρ) be a complete metric space and $A\subseteq X$. The metric space (A,ρ_A) is complete if and only if A is closed.

Proof. $\implies$: Suppose that (A,ρ_A) is complete. Let $(x_n)_n$ be a sequence in A such that $x_n\to x$. The sequence $(x_n)_n$ is basic, as convergent, in (X,ρ). Hence it is also basic in (A,ρ_A). Since (A,ρ_A) is complete, $(x_n)_n$ converges to a point $y\in A$. By uniqueness of limits, we get $x=y$. Therefore $x\in A$ and A is closed.
$\impliedby$: Suppose that A is closed. Let $(x_n)_n$ be a basic sequence in (A,ρ_A). Clearly, $(x_n)_n$ is basic in (X,ρ) and since the latter space is complete, $(x_n)_n$ converges to some point in $x\in X$. Closedness of A implies that $x\in A$, and so (A,ρ_A) is complete. □

Proposition 2.11. ***[Uniform limits of continuous functions are continuous]***
Let $f_n\colon(X,\rho)\to\mathbb{R}$ for all $n\in\mathbb{N}$ be a sequence of continuous functions and $f\colon X\to\mathbb{R}$. If $(f_n)_n$ converges uniformly to f, then f is continuous.

Proof. Fix $x \in X$ and let $(x_n)_n$ in X such that $x_n \to x$. Let $\varepsilon > 0$. By the triangular inequality, we have

$$|f(x_k) - f(x)| \leq |f(x_k) - f_n(x_k)| + |f_n(x_k) - f_n(x)| + |f_n(x) - f(x)|.$$

Since $(f_n)_n$ converges uniformly to f, there exists a $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $|f(x_k) - f_n(x_k)| < \frac{\varepsilon}{3}$ and $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$. For all $n \in \mathbb{N}$, f_n is continuous, hence there exists a $k_0 = k_0(n) \in \mathbb{N}$, such that if $k \geq k_0$, then $|f_n(x_k) - f_n(x)| < \frac{\varepsilon}{3}$. In particular, for $n \geq n_0$ and for $k \geq k_0(n)$, we get

$$|f(x_k) - f(x)| < 3\frac{\varepsilon}{3} = \varepsilon.$$

Since $(x_n)_n$ was arbitrary, f is continuous. □

Corollary 2.12. *Let (X, ρ) be a metric space. The metric space $(\mathcal{C}_b(X), d_\infty)$ is complete.*

Proof. By proposition (2.11), $\mathcal{C}_b(X)$ is a closed subset of $\mathcal{B}(X)$ and hence complete with respect to d_∞. □

2.2 Contraction Mapping Theorem

Definition 2.13. [Contractions]
Let (X, ρ) be a metric space. A mapping $T\colon X \to X$ is called a contraction if there exists a $\lambda \in (0, 1)$ such that for all $x, y \in X$

$$\rho(Tx, Ty) \leq \lambda\rho(x, y).$$

Furthermore, we say that λ is the modulus of the contraction T.

Remark 2.14. [Contractions are continuous]
Every contraction is Lipschitz continuous and hence uniformly continuous and also continuous.

Proposition 2.15. *Let (X, ρ) be a complete metric space and $T\colon X \to X$ be a contraction with modulus λ. Then for every $x \in X$ and $n \in \mathbb{N}$*

$$\rho\left(T^{n+1}x, T^n x\right) \leq \lambda^n \rho(Tx, x).$$

Proof. We fix $x \in X$ and use induction. For $n = 1$, the relation holds by contraction's definition. Let the relation hold for $n = k$. Then,

$$\rho\left(T^{k+2}x, T^{k+1}x\right) = \rho\left(T(T^{k+1}x), T(T^k x)\right) \leq \lambda\rho\left(T^{k+1}x, T^k x\right)$$

and the result follows from the induction hypothesis. □

Theorem 2.16. ***[Banach's fixed point theorem]***
Let (X, ρ) be a complete metric space. If $T\colon X \to X$ is a contraction on X with modulus λ, then T has a unique fixed point $x_0 \in X$. Moreover for every $x \in X$ we have $T^n x \to x_0$ as $n \to \infty$.

Proof. Fix $x \in X$. Firstly, we show that the sequence $(T^n x)_n$ is basic. Fix $\varepsilon > 0$. Choose $n_0 \in \mathbb{N}$ large enough, so that for all $n \geq n_0$

$$\frac{\lambda^n}{1-\lambda}\rho(Tx, x) < \varepsilon$$

holds. Let $m > n \geq n_0$. Using the triangular inequality and proposition (2.15), we get

$$\begin{aligned}
\rho(T^m x, T^n x) &\leq \sum_{j=n}^{m-1} \rho(T^{j+1}x, T^j x) \\
&\leq \sum_{j=n}^{m-1} \lambda^j \rho(Tx, x) \\
&= \lambda^n \frac{1-\lambda^{m-n}}{1-\lambda}\rho(Tx, x) \\
&\leq \lambda^n \frac{1}{1-\lambda}\rho(Tx, x) < \varepsilon.
\end{aligned}$$

Since X is complete, $(T^n x)_n$ is convergent as basic. Let x_0 be its limit.

We will show that x_0 is a fixed point of T. Note that T is continuous as a contraction. Thus, we have

$$x_0 = \lim_{n\to\infty} T^{n+1}x = \lim_{n\to\infty} T\left(T^n x\right) = T \lim_{n\to\infty} T^n x = Tx_0.$$

For uniqueness, suppose that x_1 is a fixed point of T. Then

$$0 \leq \rho(x_0, x_1) = \rho(Tx_0, Tx_1) \leq \lambda\rho(x_0, x_1),$$

which can only hold if $x_0 = x_1$, because $\lambda \in (0, 1)$. □

Theorem 2.17. ***[Blackwell's theorem]***
Let (X, ρ) be a metric space and $T\colon \mathcal{B}(X) \to \mathcal{B}(X)$. Suppose that T is increasing and there exists some $\lambda \in (0, 1)$ such that for all $f \in \mathcal{B}(X)$ and $a \geq 0$ we have

$$T(f + a) \leq Tf + \lambda a.$$

Then T is a contraction on X with modulus λ.

Proof. Let $f, g \in \mathcal{B}(X)$. Fix $x \in X$. We have

$$|f(x) - g(x)| \leq \|f - g\|_\infty,$$

hence $f(x) \leq g(x) + \|f - g\|_\infty$ and $g(x) \leq f(x) + \|f - g\|_\infty$. From the first inequality we have

$$(Tf)(x) \leq (T(g + \|f - g\|_\infty))(x) \leq (Tg)(x) + \lambda \|f - g\|_\infty,$$

hence $(Tf)(x) - (Tg)(x) \leq \lambda \|f - g\|_\infty$. Similarly, we get $(Tg)(x) - (Tf)(x) \leq \lambda \|f - g\|_\infty$ from the second inequality. Combining the two results we conclude

$$|(Tf)(x) - (Tg)(x)| \leq \lambda \|f - g\|_\infty.$$

Since x was arbitrary and $(Tf)(x) - (Tg)(x)$ is bounded, we get the result. □

2.3 Exercises

A Group

A.1. Let $(x_n)_n$ be a sequence in $(\mathbb{R}^k, d_2)$, where $k > 1$. Show that $(x_n)_n$ is basic if and only if $(x_n(j))_n$ is basic for all $j = 1, \dots, k$.

A.2. Let (X, ρ) be a complete metric space and T a contraction on X. Let A be a closed subset of X and $B \subseteq A$. Show that if $T(A) \subseteq A$, then the fixed point of T is in A. Moreover, if $T(A) \subseteq B$, then the fixed point of T is in B.

A.3. Let ρ_1, ρ_2 be two metrics on X. Suppose that there exist $a, b > 0$ such that for all $x, y \in X$ we have

$$a\rho_1(x, y) \leq \rho_2(x, y) \leq b\rho_1(x, y).$$

Show that a sequence $(x_n)_n$ is basic with respect to ρ_1 if and only if it is basic with respect to ρ_2.

B Group

B.1. Let $f_n \colon X \to \mathbb{R}$ for all $n \in \mathbb{N}$ be a sequence of functions and $f \colon X \to \mathbb{R}$. Show that if $(f_n)_n$ converges uniformly to f, then it converges pointwise.

B.2. Let $(x_n)_n$ be a sequence in the metric space (X, ρ). Show that $(x_n)_n$ is basic if and only if

$$\lim_{n\to\infty} \sup_{m\geq n} \rho(x_n, x_m) = 0.$$

B.3. Let (X, ρ) be a metric space and define

$$\mathcal{C}_b^k(X) = \{f \colon X \to \mathbb{R} \colon \quad f \text{ is } k \text{ times defferentiable with bounded derivatives}\}$$

and

$$\|f\|_k = \sup_{x\in X} \sum_{j=1}^{k} \left| f^{(j)}(x) \right|.$$

(a) Is $\|\cdot\|_\infty$ a norm on $\mathcal{C}_b^1(X)$?

(b) Show that $\|\cdot\|_k$ is a well defined norm in $\mathcal{C}_b^1(X)$.

(c) Show that the normed space $(\mathcal{C}_b^1(X), \|\cdot\|)$ is complete.

B.4. Let (X, ρ) be a metric space. Show that X is complete if and only if for all $x \in X$ and $\varepsilon > 0$, the closed ball $\bar{B}_\varepsilon(x)$ is complete.

B.5. Show that the set

$$c_{00} = \{\{x_n\}_n \subseteq \mathbb{R} \colon \quad \exists n \ \forall m \geq n \ x_m = 0\},$$

endowed with $\|\cdot\|_\infty$ is not complete.

C Group

C.1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be Lipschitz. Consider the differential equation

$$\begin{aligned} u'(t) &= f(u(t)) \qquad (t \in (0, M)) \\ \lim_{t\to 0} u(t) &= c. \end{aligned}$$

We will show that the equation above has a unique solution.

(a) Show that the operator

$$T\colon \mathcal{C}_b([0, M]) \to \mathcal{C}_b([0, M])\colon \ (Tu)(t) = c + \int_0^t f(u(s))ds$$

is well defined.

(b) Show that T is a contraction with respect to $\|\cdot\|_\infty$.

(c) Show that the fixed point of T is differentiable on $(0, M)$ and conclude that it solves the given differential equation.

Chapter 3

Compactness

3.1 Compactness in metric spaces

Definition 3.1. [Open cover]
Let (X, ρ) be a metric space. Let $\{G_i\}_{i\in I}$, where I can be uncountable, be a collection of open sets in X. We say that $\{G_i\}_i$ is an open cover of $A \subseteq X$ if $A \subseteq \cup_{i\in I} G_i$. We say that $\{G_i\}_i$ is a finite (open) cover if I is finite. For any $J \subseteq I$ such that $\{G_i\}_{i\in J}$ covers A, we say that the collection $\{G_i\}_{i\in J}$ is a sub-cover of A.

Example 3.2. We consider $(\mathbb{R}, |\cdot|)$.

1. The collection $\{(-n, n)\}_{n\in\mathbb{N}}$ is an open cover of $\mathbb{R}$. Note that there exists no finite sub-cover in this case.

2. The collection $\left\{\left(-2 + \frac{1}{n}, 2 - \frac{1}{n}\right)\right\}_{n\in\mathbb{N}}$ is an open cover of $(-2, 2)$. Also in this case, there exists no finite sub-cover of $(-2, 2)$.

Definition 3.3. [Compactness]
Let (X, ρ) be a metric space. We say that $K \subseteq X$ is compact if every open cover of K has a finite sub-cover .

Proposition 3.4. *Let (X, ρ) be a metric space and $(x_n)_n$ in X such that $x_n \to x$. Then $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact.*

Proof. Let $\{G_i\}_{i\in I}$ be an open cover. Then there exists $i_0 \in I$ such that $x \in G_{i_0}$. Since $x_n \to x$, there exists a $n_0 \in \mathbb{N}$ such that $x_n \in G_{i_0}$ for all $n \geq n_0$. Moreover, since $\{G_i\}_i$ is a cover, for each $n < n_0$, we can find $i_n \in I$ such that $x_n \in G_{i_n}$. By construction the collection $\{G_{i_k}\}_{k=0}^{n_0-1}$ is a finite sub-cover. □

Proposition 3.5. ***[Compact sets are closed]***
Let (X, ρ) be a metric space and $K \subseteq X$ be compact. Then K is closed.

Proof. Let $y \in X\backslash K$. Define $r(x) = \frac{\rho(x,y)}{2}$ for all $x \in K$. Then $\left\{B_{r(x)}(x)\right\}_{x\in K}$ is an open cover of K. By compactness, there exist $\{x_i\}_{i=1}^n \subseteq K$ such that $\left\{B_{r(x_i)}(x_i)\right\}_{i=1}^n$ is a finite sub-cover. Set $r = \min_{i=1,\dots,n}\{r(x_i)\}$. We claim that $B_r(y) \subseteq X\backslash K$. Suppose not. Then there exists some $x \in B_r(y) \cap K$. Since $\left\{B_{r(x_i)}(x_i)\right\}_{i=1}^n$ is a finite sub-cover, there exists some i_0 such that $x \in B_{r(x_{i_0})}(x_{i_0})$. Hence

$$\rho(x, y) \geq \rho(y, x_{i_0}) - \rho(x_{i_0}, x) > 2r(x_{i_0}) - r(x_{i_0}) > r,$$

which contradicts $x \in B_r(y)$. □

Proposition 3.6. ***[Compact sets are bounded]***
Let (X, ρ) be a metric space and $K \subseteq X$ be compact. Then K is bounded.

Proof. Fix $x \in K$ and consider $\{B_n(x)\}_{n\in\mathbb{N}}$. The latter is an open cover of K, hence there exist $\{n_1, \dots, n_m\} \subseteq \mathbb{N}$ such that $\{B_{n_i}(x)\}_{i=1}^m$ is a finite sub-cover. Set $N = \max_{i=1,\dots,m} n_i$. We have

$$K \subseteq \cup_{i=1}^m B_{n_i}(x) \subseteq B_N(x),$$

which completes the proof. □

Proposition 3.7. ***[Closed subsets of compact sets are compact]***
Let (X, ρ) be a metric space, $K \subseteq X$ be compact and $F \subseteq K$ be closed. Then F is compact.

Proof. Let $\{G_i\}_{i\in I}$ be an open cover of F. Then the collection $(\cup_{i\in I} G_i) \cup (X \backslash F)$ is an open cover of K. By compactness, there exist a finite $J \subseteq I$ such that $(\cup_{i\in J} G_i) \cup (X \backslash F)$ is a finite sub-cover of K. Therefore, $\{G_i\}_{i\in J}$ is finite a sub-cover of F. □

Definition 3.8. [Totally bounded sets]
Let (X, ρ) be a metric space. We say that $A \subseteq X$ is totally bounded if for every $\varepsilon > 0$ there exist $\{x_1, \dots, x_n\} \in X$ such that $A \subseteq \cup_{i=1}^n B_\varepsilon(x_i)$.

Proposition 3.9. ***[Total boundedness is stricter than boundedness]***
Let (X, ρ) be a metric space. If $A \subseteq X$ is totally bounded then it is bounded.

Proof. By total boundedness, there exist $\{x_1, \dots, x_n\} \in X$ such that $A \subseteq \cup_{i=1}^n B_1(x_i)$. Define $M = \max_{i,j=1,\dots,n} \rho(x_i, x_j)$. Let $x, y \in A$. Since $\{B_1(x_i)\}_i$ covers A, there exist i_1, i_2 such that $x \in B_1(x_{i_1})$ and $y \in B_1(x_{i_2})$. Thus

$$\begin{aligned}\rho(x, y) &\le \rho(x, x_{i_1}) + \rho(x_{i_1}, x_{i_2}) + \rho(x_{i_2}, y)\\ &\le 1 + M + 1.\end{aligned}$$

Since x, y were arbitrary, we conclude that $\operatorname{diam}(X) \le 2 + M$. □

Remark 3.10. The reverse relation of proposition (3.9) does not hold. The metric space $(\mathbb{R}, \delta)$, where δ is the discrete metric, is bounded with $\operatorname{diam}(\mathbb{R}) = 1$. However, it is not totally bounded, since it cannot be covered by a finite number of open balls with radius $\frac{1}{2}$.

Proposition 3.11. ***[Sequential characterization of total boundedness]***
Let (X, ρ) be a metric space. Then $A \subseteq X$ is totally bounded if and only if every sequence $(a_n)_n$ in A has a basic subsequence.

Proof. $\Longrightarrow$: Let $(a_n)_n$ be a sequence in A. Since A is totally bounded, there exist $\{x_1^1, \dots, x_{N_1}^1\} \in A$ such that

$$A \subseteq \cup_{i=1}^{N_1} B_1(x_i^1).$$

The collection of balls $\{B_1(x_i^1)\}_{i=1}^{N_1}$ is finite so there exists at least one of them that contains infinitely many elements of $(a_n)_n$. Without loss of generality, suppose that the last observation holds for $B_1(x_1^1)$. Let n_1 be the smallest index of the elements of $(a_n)_n$ contained in $B_1(x_1^1)$. Note that $B_1(x_1^1)$ is totally bounded as a subset of A. Hence there exist $\{x_1^2, \dots, x_{N_2}^2\} \in B_1(x_1^1)$ such that

$$B_1(x_1^1) \subseteq \cup_{i=1}^{N_2} B_{\frac{1}{2}}(x_i^2).$$

Without loss of generality assume that $B_{\frac{1}{2}}(x_1^2)$ contains infinitely many terms of $(a_n)_n$. Let n_2 be the smallest index of the elements of $(a_n)_n$ contained in $B_{\frac{1}{2}}(x_1^2)$ that is greater than n_1. Note that $a_{n_2} \in B_1(x_1^2)$. Continuing in this manner, we construct a subsequence $(a_{n_k})_k$ such that for all $k \geq m$ we have

$$a_{n_k} \in B_{\frac{1}{m}}(x_i^m).$$

Let $\varepsilon > 0$. Choose k_0 such that $\frac{1}{k_0} < \frac{\varepsilon}{2}$. Then for all $k \geq k_0$

$$a_{n_k} \in B_{\frac{1}{k_0}}(x_i^{k_0}).$$

Hence if $m, k \geq k_0$, then

$$\rho(a_{n_k}, a_{n_m}) \leq \rho(a_{n_k}, x_i^{k_0}) + \rho(x_i^{k_0}, a_{n_m}) \leq 2\frac{\varepsilon}{2} = \varepsilon.$$

$\Longleftarrow$: Suppose that A is not totally bounded. Then there exists some $\varepsilon > 0$ such that for all $N_1 \in \mathbb{N}$ and $x_1^1, \dots, x_{N_1}^1$ in A we have

$$A \setminus \cup_{i=1}^{N_1} B_\varepsilon(x_i^1) \neq \emptyset.$$

Pick some element $a_1 \in A$ and define $A_2 = A \setminus B_\varepsilon(a_1)$. Observe that A_2 is not totally bounded. If it was, we could have find a finite collection of balls $\{B_\varepsilon(x_i)\}_{i=1}^N$ that covers A_2. Then the collection of $\{B_\varepsilon(x_i)\}_{i=1}^N$ and $B_\varepsilon(a_1)$ would cover A. Pick $a_2 \in A_2$ and define $A_3 = A_2 \setminus B_\varepsilon(a_2) = A \setminus \cup_{i=1}^2 B_\varepsilon(a_i)$. Similar arguments show that A_3 is not totally bounded and we can pick $a_3 \in A_3$. Recursively, we construct a sequence $(a_n)_n$ such that

$$a_{n+1} \notin \cup_{i=1}^n B_\varepsilon(a_i).$$

Therefore $\rho(a_n, a_m) \geq \varepsilon$ for all $n \neq m$ and $(a_n)_n$ does not have a basic subsequence. □

Definition 3.12. [Sequential compactness]
Let (X, ρ) be a metric space. We say that $K \subseteq X$ is sequentially compact if for every $(x_n)_n$ in K, there exists a subsequence $(x_{n_k})_k$ that converges to some point in K.

Theorem 3.13. ***[Compactness characterizations in metric spaces]***
Let (X, ρ) be a metric space. The following are equivalent:

(i) X is compact.

(ii) Every infinite $A \subseteq X$ has at least one limit point (i.e. $A' \neq \emptyset$).

(iii) X is sequentially compact.

(iv) X is complete and totally bounded.

Proof. $(i) \implies (ii)$: Suppose there exists some infinite A such that it has no accumulation point. In other words, for all $x \in X$ there exists some $\varepsilon(x) > 0$ such that $B_{\varepsilon(x)}(x) \cap A \setminus \{x\} = \emptyset$. Note that $\left\{B_{\varepsilon(x)}(x)\right\}_{x \in X}$ is an open cover of X. Thus, we can find $x_1, \dots, x_n$ such that $\left\{B_{\varepsilon(x_i)}(x_i)\right\}_{i=1}^n$ is a finite sub-cover. This implies that

$$A \subseteq \cup_{i=1}^n \left(A \cap B_{\varepsilon(x_i)}(x_i)\right).$$

We have chosen $\left\{B_{\varepsilon(x_i)}(x_i)\right\}_{i=1}^n$ such that $B_{\varepsilon(x_i)}(x_i)\cap A\backslash\{x_i\}=\emptyset$, hence we have $A\cap B_{\varepsilon(x_i)}(x_i)\subseteq\{x_i\}$. Taking the union over all i, we then get

$$A\subseteq\cup_{i=1}^n\left(A\cap B_{\varepsilon(x_i)}(x_i)\right)\subseteq\cup_{i=1}^n\{x_i\},$$

which contradicts that A is infinite. Hence A has at least one accumulation point.

$(ii)\implies(iii)$: Let $(x_n)_n$ be a sequence in X. We will find a converging subsequence of $(x_n)_n$. Let $A=\{x_n\}_{n\in\mathbb{N}}$. If A is finite, then $(x_n)_n$ is finally constant and thus convergent. Suppose that A is infinite. By assumption, A has an accumulation point, say x. Then for all $n\in\mathbb{N}$, we have $B_{\frac{1}{n}}(x)\cap A\backslash\{x\}\neq\emptyset$. Choose an increasing sequence $(k_n)_n$ such that $x_{k_n}\in B_{\frac{1}{n}}(x)\cap A\backslash\{x\}$. Clearly, $(x_{k_n})_n$ is a subsequence of $(x_n)_n$ and by construction

$$\rho(x,x_{k_n})<\frac{1}{n}.$$

Therefore, $x_{n_k}\to x$.

$(iii)\implies(iv)$: For completeness, let $(x_n)_n$ be a basic sequence in X. Then, by sequential compactness, $(x_n)_n$ has a converging subsequence. Since every basic sequence with a convergent subsequence is convergent, $(x_n)_n$ is convergent. Since $(x_n)_n$ was arbitrary, X is complete.

We prove total boundedness by its sequential characterization. Let $(x_n)_n$ be a sequence in X. By (iii), there exists a convergent subsequence, say $(x_{k_n})_n$, of $(x_n)_n$. As convergent, $(x_{k_n})_n$ is basic. Since $(x_n)_n$ was arbitrary, X is totally bounded.

$(iv)\implies(i)$: Assume that X is complete and totally bounded, but not compact. Then there exists at least one open cover, say $\{G_i\}_{i\in I}$ that does not have a finite subcover. Since X is totally bounded, there exist $\{x_1^1,\ldots,x_{N_1}^1\}\in X$ such that

$$X\subseteq\cup_{i=1}^{N_1}B_{\frac{1}{2}}(x_i^1).$$

We claim that there exists $i_0\in\{1,\ldots,N_1\}$ such that for all $n\in\mathbb{N}$ and $i_1,\ldots,i_n\in I$ we have

$$B_{\frac{1}{2}}(x_{i_0}^1)\backslash\cup_{j=1}^n G_{i_j}\neq\emptyset.$$

Indeed, if this was not the case, then for every $i\in\{1,\ldots,N_1\}$ we could have chosen $j_1^i,\ldots,j_{n_i}^i\in I$ such that

$$B_{\frac{1}{2}}(x_i^1)\subseteq\cup_{k=1}^{n_i}G_{j_k^i}.$$

Then $\{G_{j_k^i}\}$ for $k=1,\ldots,n_i$ and $i=1,\ldots,N_1$ would have been a finite sub-cover of K. Set $x_1=x_{i_0}^1$. Since

$$B_{\frac{1}{2}}(x_1)\subseteq X,$$

there exist $\{x_1^2,\ldots,x_{N_2}^2\}\in X$ such that

$$B_{\frac{1}{2}}(x_1)\subseteq\cup_{i=1}^{N_2}B_{\frac{1}{4}}(x_i^2).$$

Without loss of generality, assume that $B_{\frac{1}{2}}(x_1)\cap B_{\frac{1}{4}}(x_i^2)\neq\emptyset$. Otherwise we can just discard the indices for which the intersection is zero. As in the previous step, we can find an $i_0\in\{1,\ldots,N_2\}$ such that for all $n\in\mathbb{N}$ and $i_1,\ldots,i_n\in I$ we have

$$B_{\frac{1}{4}}(x_{i_0}^2)\backslash\cup_{j=1}^n G_{i_j}\neq\emptyset.$$

Set $x_2 = x_{i_0}^2$. Note that $B_{\frac{1}{2}}(x_1) \cap B_{\frac{1}{4}}(x_2) \neq \emptyset$, so by picking some w in this intersection we get

$$\rho(x_1, x_2) \leq \rho(x_1, w) + \rho(w, x_2) < \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

Continuing in this manner, we construct a sequence $(x_n)_n$ that satisfies the following two conditions:

1. For all $n \in \mathbb{N}$, for all $m \in \mathbb{N}$ and for all $i_1, \ldots, i_m \in I$ we have

$$B_{\frac{1}{2^n}}(x_n) \setminus \cup_{j=1}^m G_{i_j} \neq \emptyset$$

   and

2. for all $n \in \mathbb{N}$ we have

$$\rho(x_{n+1}, x_n) < \frac{3}{2^{n+1}}.$$

Condition 2 implies that $(x_n)_n$ is basic. Indeed, let $\varepsilon > 0$ and choose n_0 such that $\frac{3}{2^{n_0}} < \varepsilon$. Then, for every $n, m \geq n_0$, we have

$$\rho(x_m, x_n) \leq \sum_{j=n}^{m-1} \rho(x_{j+1}, x_j) < \sum_{j=n}^{m-1} \frac{3}{2^{j+1}} = \frac{3}{2^n} \leq \frac{3}{2^{n_0}} < \varepsilon,$$

which proves that $(x_n)_n$ is basic. Then, completeness implies that $(x_n)_n$ is convergent. Suppose that $x_n \to x \in X$. Since $\{G_i\}_{i \in I}$ covers X, there exists some $i_0 \in I$ such that $x \in G_{i_0}$. Since G_{i_0} is open, there exists some $\delta > 0$ such that $B_\delta(x) \subseteq G_{i_0}$. Choose $n \in \mathbb{N}$, large enough, such that we have $\rho(x_n, x) < \frac{\delta}{2}$ and $\frac{1}{2^n} < \frac{\delta}{2}$. For all $w \in B_{\frac{1}{2^n}}(x_n)$ we have

$$\rho(w, x) \leq \rho(w, x_n) + \rho(x_n, x) < \frac{1}{2^n} + \frac{\delta}{2} < \delta,$$

i.e. $w \in B_\delta(x)$. Therefore,

$$B_{\frac{1}{2^n}}(x_n) \subseteq B_\delta(x) \subseteq G_{i_0},$$

which contradicts condition 1. □

Proposition 3.14. ***[Heine-Borel theorem]***
Let S be a non-empty subset of the metric space $(\mathbb{R}^n, d_2)$. Then S is compact if and only if it is closed and bounded.

Proof. $\Longrightarrow$: Compactness implies closedness (proposition (3.5)) and boundedness (proposition 3.6)).
$\Longleftarrow$: Let $(x_n)_n$ be a sequence in S. Since S is bounded, $(x_n)_n$ is bounded and it has a converging subsequence $(x_{n_k})_k$. Say that $x_{n_k} \to x$. Since S is closed, we have $x \in S$. Hence S is sequentially compact. □

3.2 Exercises

A Group

A.1. Show that (X, δ) is a compact metric space if and only if X is finite.

A.2. Let $a, b \in \mathbb{R}$ such that $a < b$. Show that $[a, b]$ is compact. Show that (a, b), $(a, b]$, $[a, b)$ and $[a, \infty)$ are not compact.

A.3. Let (X, ρ) be a metric space and $A, B \subseteq X$. Show that if A, B are compact, then $A \cup B$ is compact. Does the reverse relation hold?

B Group

B.1. Let (X, ρ) be a metric space. Show that X is complete if and only if every infinite, totally bounded subset of X has at least one accumulation point.

B.2. Let (X, ρ) be a compact metric space and $f\colon X \to \mathbb{R}$. Show that the following are equivalent.

(i) f is continuous.

(ii) The graph mapping $G_f\colon X \to X \times \mathbb{R}$: $G_f(x) = (x, f(x))$ is continuous.

(iii) The graph $G(f) = \{(x, f(x))\colon\ x \in X\}$ is compact in $X \times \mathbb{R}$.

Is it necessary for X to be compact?

B.3. Let (X, ρ) be a metric space and $A_1, \ldots, A_n$ totally bounded subsets of X. Show that their union is totally bounded.

C Group

C.1. Let S be the unit sphere in ℓ_∞. Show that S is not compact. Hint: Show that $\left\{B_{\frac{1}{2}}(x)\right\}_{x \in \ell_\infty}$ does not have a finite sub-cover.

Chapter 4

Optimization

4.1 Correspondences

Definition 4.1. [Correspondence]
Let X and Y be two non empty sets. A correspondence φ is a rule of association of elements of X with subsets of Y. We symbolize correspondences as $\varphi\colon X \twoheadrightarrow Y$. If $\varphi(x) \neq \emptyset$ for all $x \in X$, we say that φ is non-empty valued. If $\varphi(x)$ is compact/convex/closed for all $x \in X$, we say that φ is compact/convex/closed valued.

Definition 4.2. [Set distance]
Let (X, ρ) be a metric space and A, B non empty subsets of X. Let

$$d = \inf_{a \in A, b \in B} \rho(a, b).$$

We say that d is the distance of A from B and write $\rho(A, B) = d$. Let x be a point in X. We denote $\rho(A, x) = \rho(A, \{x\})$ and say that $\rho(A, x)$ is the distance of x from A.

Lemma 4.3. *Let (X, ρ) be a metric space, F a closed subset of X and K a compact subset of X. If $K \cap F = \emptyset$, then $\rho(F, K) > 0$.*

Proof. Suppose that $\rho(F, K) = 0$. Then we can choose sequences $(x_n)_n$ in F and $(y_n)_n$ in K such that $\rho(x_n, y_n) \to 0$. Since K is compact, there exists a convergent subsequence $(y_{n_k})_k$ with $y_{n_k} \to y \in K$. Hence, we also have $x_{n_k} \to y$. Since F is closed, we get $y \in F$. Therefore $K \cap F \neq \emptyset$. $\square$

Definition 4.4. [Hemicontinuity]
Let $\varphi\colon (X, \rho) \to (Y, d)$ be a correspondence between two metric spaces. We say that φ is

(i) lower hemicontinuous at $x \in X$ if for all $y \in Y$ and $\varepsilon > 0$ such that $B_\varepsilon(y) \cap \varphi(x) \neq \emptyset$ there exists $\delta > 0$ such that for all $z \in B_\delta(x)$ we have $B_\varepsilon(y) \cap \varphi(z) \neq \emptyset$ and

(ii) upper hemicontinuous at $x \in X$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $z \in B_\delta(x)$ we have $\varphi(z) \subseteq \{y \in Y : d(\varphi(x), y) < \varepsilon\}$.

As with functions, we say φ is upper/lower hemicontinuous on X, if it is upper/lower hemicontinuous at every point of X.

Proposition 4.5. ***[Sequential lower hemicontinuity]***
Let $\varphi\colon (X, \rho) \to (Y, d)$ be a non empty valued correspondence between two metric spaces. Then φ is lower hemicontinuous at $x \in X$ if and only if for every $y \in \varphi(x)$ and every sequence $(x_n)_n$ in X such that $x_n \to x$, there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ and elements $y_k \in \varphi(x_{n_k})$ such that $y_k \to y$.

Proof. $\Longrightarrow$: Let $y \in \varphi(x)$ and $(x_n)_n$ in X such that $x_n \to x$. Note that $y \in B_1(y) \cap \varphi(x)$, so there exists $\delta_1 > 0$ such that for all $z \in B_{\delta_1}(x)$ we have $B_1(y) \cap \varphi(z) \neq \emptyset$. Since $x_n \to x$, there exists $n_1 = n(\delta_1) \in \mathbb{N}$ such that $x_{n_1} \in B_{\delta_1}(x)$. Pick $y_1 \in B_1(y) \cap \varphi(x_{n_1})$. We also have $y \in B_{\frac{1}{2}}(y) \cap \varphi(x)$, so there exists $\delta_2 > 0$ such that for all $z \in B_{\delta_2}(x)$ we have $B_{\frac{1}{2}}(y) \cap \varphi(z) \neq \emptyset$. There exists $n_2 = n(\delta_2) \in \mathbb{N}$ such that $n_2 > n_1$ and $x_{n_2} \in B_{\delta_2}(x)$. Pick $y_2 \in B_{\frac{1}{2}}(y) \cap \varphi(x_{n_2})$. Recursively we find a subsequence (x_{n_k}) and elements $y_k \in B_{\frac{1}{k}}(y) \cap \varphi(x_{n_k})$. Hence $y_k \in \varphi(x_{n_k})$ and $y_k \to y$.
$\Longleftarrow$: Towards contradiction, suppose that φ is not lower hemicontinuous. Then there exist $y \in X$ and $\varepsilon > 0$ such that $B_\varepsilon(y) \cap \varphi(x) \neq \emptyset$ and for all $\delta > 0$ there is some $z \in B_\delta(x)$ such that $B_\varepsilon(y) \cap \varphi(z) = \emptyset$. Picking $\delta = \frac{1}{n}$ we can recursively construct a sequence $(x_n)_n$ such that $x_n \in B_{\frac{1}{n}}(x)$ and $B_\varepsilon(y) \cap \varphi(x_n) = \emptyset$. Since $x_n \to x$, there exist a subsequence $(x_{n_k})_k$ and $y_k \in \varphi(x_{n_k})$ such that $y_k \to y$. Hence, we can find $k_0 \in \mathbb{N}$ such that $y_{k_0} \in B_\varepsilon(y)$, which implies that $y_{k_0} \in B_\varepsilon(y) \cap \varphi(x_{n_{k_0}})$. The last implication contradicts the construction of $(x_n)_n$, thus φ is lower hemicontinuous. $\square$

Proposition 4.6. ***[Sequential upper hemicontinuity]***
Let $\varphi\colon (X,\rho) \to (Y,d)$ be a non empty, compact valued correspondence between two metric spaces. Then φ is upper hemicontinuous at $x \in X$ if and only if for every every sequence $(x_n)_n$ such that $x_n \to x$ and for all $(y_n)_n$ such that $y_n \in \varphi(x_n)$ there exists a subsequence $(y_{n_k})_k$ which converges at some point $y \in \varphi(x)$.

Proof. $\Longrightarrow$: Let $x_n \to x$ and $y_n \in \varphi(x_n)$ for each $n \in \mathbb{N}$. Towards contradiction, suppose that no subsequence of $(y_n)_n$ converges at a point in $\varphi(x)$. Let $y \in \varphi(x)$. Then there exists some $r(y) > 0$ and $n(y) \in \mathbb{N}$ such that $y_n \notin B_{r(y)}(y)$ for all $n \geq n(y)$. Note that the collection $\{B_{r(y)}(y)\}_{y \in \varphi(x)}$ is an open cover of $\varphi(x)$. Since φ is compact valued, we can find $y^1, \dots, y^m$ such that $\{B_{r(y^i)}(y^i)\}_{i=1}^m$ is a subcover. Define $n_0 = \max_{i=1,\dots,m} n(y^i)$. Then for all $n \geq n_0$ we have $y_n \notin \cup_{i=1}^m B_{r(y^i)}(y^i)$ and thus $y_n \notin \varphi(x)$. Let $F = Y \setminus \cup_{i=1}^m B_{r(y^i)}(y^i)$. Note that F is closed, $\varphi(x)$ compact and they are disjoint, hence $d(\varphi(x), F) > 0$. Let $0 < \varepsilon < d(\varphi(x), F)$ and note that $d(\varphi(x), y_n) > \varepsilon$ because $y_n \in F$. By upper hemicontinuity, there exists some $\delta > 0$ such that if $z \in B_\delta(x)$ then $\varphi(z) \subseteq \{y \in Y \colon d(\varphi(x), y) < \varepsilon\}$. Since $x_n \to x$, we can find $n \geq n_0$ such that $x_n \in B_\delta(x)$. But then $y_n \in \varphi(x_n) \subseteq \{y \in Y \colon d(\varphi(x), y) < \varepsilon\}$, i.e. $d(\varphi(x), y_n) < \varepsilon$, which is a contradiction.
$\Longleftarrow$: We prove by contradiction. Suppose that φ is not upper hemicontinuous at x. Then there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ there is some $x_n \in B_{\frac{1}{n}}(x)$ and $y_n \in \varphi(x_n)$ such that $d(\varphi(x), y_n) \geq \varepsilon$. Since $x_n \to x$, there exists a subsequence $(y_{n_k})_k$ such that $y_{n_k} \to y \in \varphi(x)$. On the other hand, we have

$$d(\varphi(x), y) = \lim_k d(\varphi(x), y_{n_k}) \geq \varepsilon,$$

hence $y \notin \varphi(x)$. $\square$

Definition 4.7. [Continuity]
A correspondence $\varphi\colon (X,\rho) \to (Y,d)$ is said to be continuous if it is both upper and lower hemicontinuous.

4.2 Theorem of the maximum

Missing 4.8. For the amusement of the reader.

Proposition 4.9. ***[Extreme value theorem]***
Let (X,ρ) be a compact metric space and $f\colon X \to \mathbb{R}$ a continuous function. Then the maximum and minimum of f are attained in X.

Proof. We only prove the statement for the maximum here. Compactness of X and continuity of f imply that $f(X)$ is compact. Indeed, let $(G_i)_{i\in I}$ be an open cover of $f(X)$. Then

$$X \subseteq f^{-1}(\cup_{i\in I} G_i) = \cup_{i\in I} f^{-1}(G_i).$$

Since f is continuous, $f^{-1}(G_i)$ is open for all $i \in I$. To see this, fix $i_0 \in I$ and $x \in f^{-1}(G_{i_0})$. Let $(x_n)_n$ be a sequence in X that converges to x. Then, we have $f(x_n) \to f(x) \in G_{i_0}$. Since G_{i_0} is open, we have $f(x_n) \in G_{i_0}$ finally. Equivalently, we have $x_n \in f^{-1}(G_{i_0})$ finally, which proves that $f^{-1}(G_{i_0})$ is open. Therefore, $\{f^{-1}(G_i)\}_{i\in I}$ is an open cover of X. Since X is compact, there exists some finite $J \subseteq I$ such that $(f^{-1}(G_i))_{i\in J}$ covers X. Hence

$$f(X) \subseteq f(\cup_{i\in J} f^{-1}(G_i)) = \cup_{i\in J} G_i$$

and so $f(X)$ is compact. Compactness of $f(X)$ implies that it is complete and totally bounded. As totally bounded, $f(X)$ is bounded so its supremum exists. By completeness, we have $\sup f(X) \in f(X)$. Hence there exists some $x_0 \in X$ such that $f(x_0) = \sup f(X)$. □

Theorem 4.10. ***[Theorem of the maximum]***
Let (X, ρ) and (Y, d) be two metric spaces. Let $f : X \times Y \to \mathbb{R}$ be a continuous function and $\varphi : X \to Y$ a compact valued and continuous correspondence. Then, the function

$$v(x) = \sup_{y\in\varphi(x)} f(x, y)$$

is continuous and the correspondence of maximizers

$$g(x) = \{y \in \varphi(x) : v(x) = f(x, y)\}$$

is non empty, compact valued and upper hemicontinuous.

Proof. Fix $x \in X$. Then $\varphi(x)$ is compact and by the extreme value theorem, the maximum of $y \mapsto f(x, y)$ is attained in $\varphi(x)$. Hence v is well defined and $g(x)$ is not empty. To show that $g(x)$ is compact, it suffices to show that it is closed, because $g(x) \subseteq \varphi(x)$. Let $(y_n)_n$ be a sequence in $g(x)$ such that $y_n \to y$. Since $\varphi(x)$ is closed as compact, we have $y \in \varphi(x)$. Moreover, we have $v(x) = f(x, y_n)$ and by continuity of f we also get $v(x) = f(x, y)$. By definition, $y \in g(x)$ and $g(x)$ is closed.

We will show that g is upper hemicontinuous at x. Let $(x_n)_n$ such that $x_n \to x$ and $(y_n)_n$ such that $y_n \in g(x_n)$. Since φ is upper hemicontinuous, as continuous, and compact valued there exists a subsequence $(y_{n_k})_k$ such that $y_{n_k} \to y$ and $y \in \varphi(x)$. Let $z \in \varphi(x)$. By lower hemicontinuity of φ, there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ and elements $z_k \in \varphi(x_{n_k})$ such that $z_k \to z$. Observe that $y_{n_k} \in g(x_{n_k})$ implies that

$$f(x_{n_k}, y_{n_k}) \geq f(x_{n_k}, z_{n_k}) \qquad \forall k \in \mathbb{N}.$$

Then continuity of f implies that $f(x, y) \geq f(x, z)$. The last relation holds for arbitrary $z \in \varphi(x)$, hence $y \in g(x)$ and g is upper hemicontinuous at x.

Lastly, we show that v is continuous at x. Let $(x_n)_n$ be a sequence in X such that $x_n \to x$. Let $(y_n)_n$ such that $y_n \in g(x_n)$. Pick a subsequence $(x_{n_k})_k$ such that

$$\limsup_{n\to\infty} v(x_n) = \lim_{k\to\infty} v(x_{n_k}).$$

Since g is upper hemicontinuous at x, there exists a subsequence $(y_{n_{k_l}})_l$ such that $y_{n_{k_l}} \to y$ for some $y \in g(x)$. Hence

$$\limsup_{n\to\infty} v(x_n) = \lim_{k\to\infty} v(x_{n_k}) = \lim_{l\to\infty} f(x_{n_{k_l}}, y_{n_{k_l}}) = f(x,y) = v(x).$$

Repeating the argument for a subsequence $(x_{n_k})_k$ such that

$$\liminf_{n\to\infty} v(x_n) = \lim_{k\to\infty} v(x_{n_k}).$$

we also get

$$\liminf_{n\to\infty} v(x_n) = v(x).$$

Hence $v(x_n) \to v(x)$ and since $(x_n)_n$ was arbitrary, v is continuous at x. □

4.3 Exercises

A Group

A.1. Let (X,ρ) be a compact metric space and $f\colon X\to\mathbb{R}$ a continuous function. Show that the minimum of f is attained in X.

A.2. Let $\varphi\colon (X,\rho)\to(Y,d)$ be a compact valued and upper hemicontinuous correspondence between metric spaces. Show that if X is compact, then

$$G(\varphi)=\{(x,y)\in X\times Y\colon\ y\in\varphi(x)\}$$

is compact.

B Group

B.1. Let $\varphi\colon (X,\rho)\to(Y,d)$ be a correspondence between metric spaces. We say that φ is single valued if $\varphi(x)$ is singleton. Suppose that φ is single valued and denote $\{y_x\}=\varphi(x)$. Define

$$f\colon X\to Y\colon f(x)=y_x.$$

Show that

(a) if φ is upper hemicontinuous, then f is continuous and

(b) if φ is lower hemicontinuous, then f is continuous.

B.2. Let $\varphi,\psi\colon (X,\rho)\to(Y,d)$ be two correspondences between metric spaces. Define

$$\varphi\cup\psi\colon X\to Y\colon (\varphi\cup\psi)(x)=\varphi(x)\cup\psi(x).$$

Show that

(a) if φ,ψ are compact valued and upper hemicontinuous, then so is $\varphi\cup\psi$ and

(b) if φ,ψ are lower hemicontinuous, then so is $\varphi\cup\psi$

B.3. Let $f,g\colon (X,\rho)\to\mathbb{R}$ be two continuous functions such that $g\le f$. Let

$$\varphi\colon X\to\mathbb{R}\colon \varphi(x)=[g(x),f(x)].$$

Show that φ is continuous.

B.4. Show that

$$d(f,g)=\int_a^b |f(x)-g(x)|\,dx$$

defines a metric on $\mathcal{C}([a,b])$.